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Homogeneous prime ideals and graded modules fitting into long Bourbaki sequences

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Abstract

For a finitely generated torsion-free graded module over a polynomial ring, there exists a homogeneous prime ideal fitting into a long Bourbaki sequence if and only if the given module is reflexive. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let p, r be integers with $2 \leq p \leq r-2$. Given a homogeneous ideal I of height p in a polynomial ring $R := k[x_1, \ldots, x_r]$, there is a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq p-1$ that fits into an exact sequence of the form

$$(*) \quad 0 \to S_{p-1} \to S_{p-2} \to \cdots \to S_1 \to S_0 \oplus M \to I(c) \to 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules (see e.g. [2,12]). Conversely, as proved in our previous paper [3], given a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}_R^i(M, R) = 0$ for $1 \leq i \leq p-1$, there is a homogeneous ideal I of height p in R fitting into an exact sequence of the above form. But, unfortunately, the residue

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class rings defined by the ideals constructed by our method of [3] are not even reduced in general.

When does there exist a prime I fitting into $(*)$? The aim of this paper is to give an answer to this question. In fact, we will prove that there is a homogeneous prime ideal I of height p fitting into an exact sequence of the form $(*)$ for some c and $S_0, \dots, S_{p-1}$, if a given module M as above is reflexive (see Theorem 2.8). Since M is reflexive if R/I is equidimensional by the local version of [12, Corollary 1.20], it turns out that the reflexiveness of M is equivalent to the existence of such a prime I. This theorem is known well for the classical case where $p=2$ and the coherent sheaf on Proj$R$ that M defines is locally free (see e.g. [7,9]).

To prove our theorem, as in the proof of the main theorem of [3], we make full use of the minimal free complex $F_\bullet$ bounded on both sides with differentials $\partial_\bullet^F$ defined by the conditions $M = \mathrm{Coker}(\partial_1^F)$, $H_i(F_\bullet)=0$ for $i>0$, and $H^i(F_\bullet^\vee)=0$ for $i \le 0$. The way we use it in the present paper, however, is very different from that in our previous one. This time, we make a kind of decomposition of $F_\bullet$ regarding its homologies as in [1], in order to construct a finitely generated torsion-free module $\tilde{M}$ over the integral normal domain $A := R/(f_1, \dots, f_{p-2})$ defined by a homogeneous R-regular sequence $f_1, \dots, f_{p-2}$ such that there is a homomorphism $\varphi : M \to \tilde{M}$ over R inducing an isomorphism $H_{\mathfrak{m}}^i(\varphi) : H_{\mathfrak{m}}^i(M) \to H_{\mathfrak{m}}^i(\tilde{M})$ for all $0 \le i < r-p+2$ (see Theorems 1.11 and 2.2). Then, applying to the A-module $\tilde{M}$ the well-known determinantal method for constructing two-codimensional subschemes, we obtain our main results. The same method has already been proposed in [11] to treat the simplest case where $F_\bullet$ is the direct sum of the free complexes giving the minimal free resolutions of graded modules of finite length over R.

This paper is organized as follows.

Section 1 consists of technical arguments. In this section, we show that for the complex $F_\bullet$ mentioned above, there is a homogeneous R-regular sequence $f_1, \dots, f_{p-2}$, minimal complex $\tilde{F}_\bullet$ of finitely generated graded free modules over $A := R/(f_1, \dots, f_{p-2})$ with $\tilde{F}_i = 0$ for all $i \ll 0$, and a chain map $\tau_\bullet : F_\bullet \to \tilde{F}_\bullet$ over A such that the canonical homomorphisms $H_{\mathfrak{m}}^i(\tau_0) : H_{\mathfrak{m}}^i(\mathrm{Coker}(\partial_1^F)) \to H_{\mathfrak{m}}^i(\mathrm{Coker}(\partial_1^{\tilde{F}}))$ $(0 \le i < r-p+2)$ and $H_i(\tau_\bullet) : H_i(F_\bullet) \to H_i(\tilde{F}_\bullet)$ $(i \in \mathbf{Z})$ are isomorphisms.

In Section 2, using the results of Section 1, we first show the existence of $\tilde{M}$ and φ described above. Then it is shown that there exists a homogeneous prime ideal I fitting into $(*)$ more or less in a standard manner.

1. Homology preserving descent of free complexes

Let $R := k[x_1, \dots, x_r]$ be a polynomial ring in r indeterminates $x_1, \dots, x_r$ over an infinite field k, $\mathfrak{m} := \bigoplus_{i>0} [R]_i$ the irrelevant maximal ideal in R, and A the graded residue class ring of R defined by a homogeneous ideal in R. We assume that A is Cohen–Macaulay. For a complex $C_\bullet$ of finitely generated graded free modules over A, we say that $C_\bullet$ is *minimal* if $\mathrm{Im}(\partial_i^C) \subset \mathfrak{m}C_{i-1}$ for all $i \in \mathbf{Z}$, where ∂_i^C $(i \in \mathbf{Z})$ are the differentials of $C_\bullet$. If there are a minimal complex $C'_\bullet$ and a split exact complex $C''_\bullet$, of finitely generated graded free modules over A, such that $C_\bullet = C'_\bullet \oplus C''_\bullet$, then we will denote $C'_\bullet$ (resp. $C''_\bullet$) by $\min(C_\bullet)_\bullet$ (resp. $\mathrm{se}(C_\bullet)_\bullet$). Further, in this case, $\min(C_\bullet)_\bullet$ (resp. $\mathrm{se}(C_\bullet)_\bullet$) will be called the *minimal* (resp. *split exact*) *part* of $C_\bullet$ (see [1, (1.1) and (1.2)]). Given a chain map $\mu_\bullet : C_\bullet \to D_\bullet$ of complexes, its mapping cone will be denoted by $\mathrm{con}(\mu_\bullet)_\bullet$.

Definition 1.1. Let $L_\bullet$ be a complex of finitely generated graded free modules over A, $\mathfrak{a}$ a homogeneous ideal in A, and m an integer. We say that a subcomplex $L'_\bullet$ of $L_\bullet$ is a quasi-direct summand of $L_\bullet$ up to $(\mathfrak{a}, m)$ if it satisfies the following conditions:

(i) $L'_\bullet$ and $L''_\bullet := L_\bullet / L'_\bullet$ are free complexes.
(ii) There is a chain map $\mu_\bullet : L''_\bullet \to L'[-1]_\bullet$ satisfying $\mathrm{Im}(\mu_i) \subset \mathfrak{a} L'_{i-1}$ for all $i \le m$ such that $L_\bullet = \mathrm{con}(\mu_\bullet)_\bullet$.

Remark 1.2. Conditions (i) and (ii) in the above definition are equivalent to the existence of graded free modules L''_i and homomorphisms $\partial''_i : L''_i \to L''_{i-1}$ and $\mu_i : L''_i \to L'_{i-1}$ ($i \in \mathbf{Z}$) such that

$$L_i = L'_i \oplus L''_i, \quad \partial^L_i := \begin{pmatrix} \partial^{L'}_i & (-1)^i \mu_i \\ 0 & \partial''_i \end{pmatrix} : L_i = L'_i \oplus L''_i \to L'_{i-1} \oplus L''_{i-1} = L_{i-1}$$

$$\text{for all } i \in \mathbf{Z} \quad \text{and} \quad \mathrm{Im}(\mu_i) \subset \mathfrak{a} L'_{i-1} \quad \text{for all } i \le m.$$

Lemma 1.3. *Let P and Q be finitely generated graded free modules over A and let $\partial : P \to Q$ be a homomorphism over A. Let further $\mathfrak{a} \subset A$ be a homogeneous ideal, $l \ge 0$ an integer, f a homogeneous element of $\mathfrak{a}^l$, $\bar{P} := P/fP$, $\bar{Q} := Q/fQ$, and $\bar{\partial} : \bar{P} \to \bar{Q}$ the homomorphism induced from ∂. Suppose that there is an integer $s' \ge 0$ such that $\mathrm{Im}(\partial) \cap \mathfrak{a}^s Q \subset \mathfrak{a}^{s-s'} \mathrm{Im}(\partial)$ for all $s \ge s'$. Then, $\mathrm{Im}(\bar{\partial}) \cap \mathfrak{a}^s \bar{Q} \subset \mathfrak{a}^{s-s'} \mathrm{Im}(\bar{\partial})$ for all s with $s' \le s \le l$.*

Proof. Suppose $s' \le s \le l$. Then, since $f \in \mathfrak{a}^l \subset \mathfrak{a}^s$, we have

$$\begin{aligned} &(\mathrm{Im}(\partial) + fQ) \cap (\mathfrak{a}^s Q + fQ) \\ &\quad \subset \mathrm{Im}(\partial) \cap (\mathfrak{a}^s Q + fQ) + fQ \\ &\quad \subset \mathrm{Im}(\partial) \cap \mathfrak{a}^s Q + fQ \subset \mathfrak{a}^{s-s'} \mathrm{Im}(\partial) + fQ \end{aligned}$$

by hypothesis. Hence, $\mathrm{Im}(\bar{\partial}) \cap \mathfrak{a}^s \bar{Q} \subset \mathfrak{a}^{s-s'} \mathrm{Im}(\bar{\partial})$ for all s with $s' \le s \le l$. □

Lemma 1.4. *Let V be a finitely generated graded module over A,*

$$(1.4.1) \qquad \cdots \to P_2 \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\partial^P} V \to 0$$

a minimal free resolution of V over A, f a homogeneous A-regular element of A annihilating V, and $\bar{P}_\bullet$ the complex $P_\bullet / fP_\bullet$. Denote by $e_1, \ldots, e_t$ the homogeneous *free bases of* P_0. *For each* $1 \le i \le t$, *let* g_i *be a homogeneous element of* $(\partial_1^P)^{-1}(fP_0)$ *such that* $\partial_1^P(g_i) = fe_i$. *Then there is an isomorphism* $\bar{\varepsilon}: V(-\deg(f)) \to H_1(\bar{P}_\bullet)$ *such that* $\bar{\varepsilon}(\delta^P(e_i)) = \bar{g}_i$ for all $1 \le i \le t$, *where* $\bar{g}_i$ *denotes the element of* $H_1(\bar{P}_\bullet)$ *represented by* g_i.

Proof. Let $K(f;A)_\bullet$ be the Koszul complex of f with respect to A. Then, there is a composition of isomorphisms

$$\begin{aligned} V(-\deg f) &= H_1(H_0(P_\bullet) \otimes_A K(f;A)_\bullet) \xrightarrow{\sim} \mathrm{Tor}_1^A(V, A/(f)) \\ &\cong H_1(P_\bullet \otimes_A K(f;A)_\bullet) \xrightarrow{\sim} H_1(P_\bullet \otimes_A H_0(K(f;A)_\bullet)) = H_1(\bar{P}_\bullet), \end{aligned}$$

which is nothing but $\bar{\varepsilon}$. □

Lemma 1.5. *Let* V *be a finitely generated graded module over* A,

$$\cdots \to P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\delta^P} V \to 0$$

a minimal free resolution of V *over* A, *and* $\mathfrak{a}$ *a homogeneous ideal in* A *annihilating* V. *Let further* $n \ge 0$ *and* m *be integers. Then, there is an integer* l *such that, for an arbitrary homogeneous* A*-regular element* f *of* $\mathfrak{a}^l$, *there is a free resolution*

$$\cdots \to \tilde{P}_2 \to \tilde{P}_1 \to \tilde{P}_0 \to V \to 0$$

of V *over* $\bar{A} := A/(f)$ *such that*

(i) $\bar{P}_\bullet := P_\bullet / fP_\bullet$ *is a quasi-direct summand of* $\tilde{P}_\bullet$ *up to* $(\mathfrak{a}^n, m)$, *where* $P_i = 0$ *and* $\tilde{P}_i = 0$ *for* $i < 0$,

(ii) *the canonical homomorphism* $H_i(\alpha_\bullet): H_i(P_\bullet) \to H_i(\tilde{P}_\bullet)$, *induced from the chain map* $\alpha_\bullet: P_\bullet \to \tilde{P}_\bullet$ *over* A *obtained by composing the natural surjection* $P_\bullet \twoheadrightarrow \bar{P}_\bullet$ *and the injection* $\bar{P}_\bullet \hookrightarrow \tilde{P}_\bullet$, *is an isomorphism for all* $i \in \mathbf{Z}$.

Proof. Let $P_i = 0$ for $i < 0$. It is enough to consider the case where m is large. By Artin–Rees lemma, there is an integer $s_i \ge 0$ for each $2 \le i \le m-1$ such that

$$\mathrm{Im}(\partial_i^P) \cap \mathfrak{a}^s P_{i-1} \subset \mathfrak{a}^{s-s_i}\, \mathrm{Im}(\partial_i^P) \quad \text{for } s \ge s_i.$$

Put $l := n + 1 + \sum_{j=2}^{m-1} s_j$ and let f be a homogeneous A-regular element of $\mathfrak{a}^l$ of degree, say u. Then

$$\mathrm{Im}(\partial_i^{\bar{P}}) \cap \mathfrak{a}^s \bar{P}_{i-1} \subset \mathfrak{a}^{s-s_i} \mathrm{Im}(\partial_i^{\bar{P}}) \quad \text{for all } s,\ i \text{ with } s_i \le s \le l,\ 2 \le i \le m-1. \tag{1.5.1}$$

by Lemma 1.3. Let

$$\cdots \to \tilde{\tilde{P}}_2 \xrightarrow{\partial_2^{\tilde{\tilde{P}}}} \tilde{\tilde{P}}_1 \xrightarrow{\partial_1^{\tilde{\tilde{P}}}} \tilde{\tilde{P}}_0 = \bar{P}_0 \xrightarrow{\delta^{\tilde{\tilde{P}}} = \delta^P \otimes \bar{A}} V \to 0$$

be a minimal free resolution of V over $\bar{A}$ and $\tilde{\tilde{P}}_i := 0$ for $i < 0$. We construct a chain map $\mu_\bullet: \tilde{\tilde{P}}[-2]_\bullet(-u) \to \bar{P}[-1]$ in the following manner. First, with the notation of Lemma 1.4, choose a homogeneous element $g_i \in (\partial_1^P)^{-1}(fP_0)$ such that $\partial_1^P(g_i) = fe_i$ for each $1 \le i \le t$. Since $f \in \mathfrak{a}^l$ and $\mathfrak{a}P_0 \subset \mathrm{Im}(\partial_1^P)$, we may assume with no loss of generality that $g_i \in \mathfrak{a}^{l-1}P_1$. Let $\bar{\delta} : \mathrm{Ker}(\partial_1^{\bar{P}}) \to H_1(\bar{P}_\bullet)$ be the canonical surjection. By Lemma 1.4, there is an isomorphism $\bar{\varepsilon} : V(-u) \to H_1(\bar{P}_\bullet)$ over $\bar{A}$ such that $\bar{\varepsilon}(\delta^P(e_i))= \bar{g}_i$ for all $1 \leq i \leq t$. Let $\mu_i = 0$ for $i < 2$ and let $\mu_2 : \tilde{\tilde{P}}_0(-u) \to \mathfrak{a}^{l-1}\bar{P}_1 \cap \mathrm{Ker}\,(\partial_1^{\bar{P}}) \subset \bar{P}_1$ be the homomorphism such that $\mu_2(e_i \otimes \bar{A}) = g_i \pmod{fP_1}$ for all $1 \leq i \leq t$. Then $\bar{\varepsilon} \circ \delta^{\tilde{\tilde{P}}} = \bar{\delta} \circ \mu_2$ and $\mathrm{Im}(\mu_2) \subset \mathfrak{a}^{l-1}\bar{P}_1$. Since $\bar{\delta} \circ \mu_2 \circ \partial_1^{\tilde{\tilde{P}}} = \bar{\varepsilon} \circ \delta^{\tilde{\tilde{P}}} \circ \partial_1^{\tilde{\tilde{P}}} = 0$ and since $\mathrm{Ker}(\bar{\delta}) = \mathrm{Im}(\partial_2^{\bar{P}})$, the image of $\mu_2 \circ \partial_1^{\tilde{\tilde{P}}}$ is contained in $\mathrm{Im}(\partial_2^{\bar{P}})$ as well as in $\mathfrak{a}^{l-1}\bar{P}_1$. Hence, $\mathrm{Im}(\mu_2 \circ \partial_1^{\tilde{\tilde{P}}}) \subset \mathfrak{a}^{l-1-s_2}\,\mathrm{Im}(\partial_2^{\bar{P}})$ by (1.5.1). There is, therefore, a homomorphism $\mu_3 : \tilde{\tilde{P}}_1(-u) \to \bar{P}_2$ satisfying $\partial_2^{\bar{P}} \circ \mu_3 = \mu_2 \circ \partial_1^{\tilde{\tilde{P}}}$ such that $\mathrm{Im}(\mu_3) \subset \mathfrak{a}^{l-1-s_2}\bar{P}_2$. Now,

$$H_i(\bar{P}_\bullet) = \mathrm{Tor}_i^R(V, A/(f)) = 0 \quad \text{for } i \geq 2$$

since f is A-regular. In the same way as above, we see that

$$\mathrm{Im}(\mu_3 \circ \partial_2^{\tilde{\tilde{P}}}) \subset \mathrm{Ker}(\partial_2^{\bar{P}}) \cap \mathfrak{a}^{l-1-s_2}\bar{P}_2 = \mathrm{Im}(\partial_3^{\bar{P}}) \cap \mathfrak{a}^{l-1-s_2}\bar{P}_2 \subset \mathfrak{a}^{l-1-s_2-s_3}\,\mathrm{Im}(\partial_3^{\bar{P}}).$$

Hence, there is a homomorphism $\mu_4 : \tilde{\tilde{P}}_2(-u) \to \bar{P}_3$ satisfying $\partial_3^{\bar{P}} \circ \mu_4 = \mu_3 \circ \partial_2^{\tilde{\tilde{P}}}$ such that $\mathrm{Im}(\mu_4) \subset \mathfrak{a}^{l-1-s_2-s_3}\bar{P}_3$. We can continue this procedure to construct μ_i $(i \geq 4)$ successively so that $\mathrm{Im}(\mu_i) \subset \mathfrak{a}^{l-1-\sum_{j=2}^{i-1} s_j}\bar{P}_{i-1}$ for all $4 \leq i \leq m$ by the help of (1.5.1). Since $l = n + 1 + \sum_{j=2}^{m-1} s_j$ and $\mu_i = 0$ for $i < 2$, we have $\mathrm{Im}(\mu_i) \subset \mathfrak{a}^n\bar{P}_{i-1}$ for all $i \leq m$. Denote by $\tilde{P}_\bullet$ be the mapping cone of $\mu_\bullet$. Let $\alpha'_\bullet : \bar{P}_\bullet \to \tilde{P}_\bullet$ be the canonical injection, $\alpha''_\bullet : P_\bullet \to \bar{P}_\bullet$ the natural surjection, and $\alpha_\bullet := \alpha'_\bullet \circ \alpha''_\bullet$. By what we have done, there is an exact sequence

$$0 \to \bar{P}_\bullet \xrightarrow{\alpha'_\bullet} \tilde{P}_\bullet \to \tilde{\tilde{P}}[-2]_\bullet(-u) \to 0,$$

and $\bar{P}_\bullet$ is a quasi-direct summand of $\tilde{P}_\bullet$ up to $(\mathfrak{a}^n, m)$ with $\tilde{P}_i = 0$ for $i < 0$. Moreover by the long exact sequence

$$\cdots \to H_{i-1}(\tilde{\tilde{P}}_\bullet)(-u) \xrightarrow{H_{i+1}(\mu_\bullet)} H_i(\bar{P}_\bullet) \xrightarrow{H_i(\alpha'_\bullet)} H_i\tilde{P}_\bullet \to H_{i-2}(\tilde{\tilde{P}}_\bullet)(-u) \xrightarrow{H_i(\mu_\bullet)} \cdots,$$

we find that $H_0(\alpha'_\bullet)$ is an isomorphism with $H_0(\tilde{P}_\bullet) \cong H_0(\bar{P}_\bullet) = V$ and that $H_i(\tilde{P}_\bullet) = 0$ for all $i > 0$, since the connecting homomorphism $H_2(\mu_\bullet)$ coincides with $\bar{\varepsilon}$. Hence the complex $\tilde{P}_\bullet$ gives a free resolution of V. On the other hand, $H_0(\alpha''_\bullet)$ is also an isomorphism, since $f \in \mathfrak{a} \subset \mathrm{ann}(V)$. Hence $H_0(\alpha_\bullet)$ is an isomorphism. In addition, since $H_i(P_\bullet) = H_i(\tilde{P}_\bullet) = 0$ for all $i \neq 0$, it is clear that $H_i(\alpha_\bullet)$ $(i \neq 0)$ are isomorphisms. Thus, the conditions (i) and (ii) hold for $\tilde{P}_\bullet$. □

Lemma 1.6. *Let a, l, m, n, s_i $(a+2 \leq i \leq m)$ be integers with $n \geq 0$, $m \geq a+1$, $0 \leq s_i \leq l$ $(a+2 \leq i \leq m)$ and $\mathfrak{a}$ be a homogeneous ideal in A. Let further $L_\bullet$, $L'_\bullet$, $G_\bullet$, and $G'_\bullet$ be complexes of finitely generated graded free modules over A such that $L'_\bullet$ (resp. $G'_\bullet$) is a quasi-direct summand of $L_\bullet$ (resp. $G_\bullet$) up to $(\mathfrak{a}^l, m)$ (resp. $(\mathfrak{a}^l, m+1)$). Suppose $G_i = 0$ for $i < a+1$,* $H_i(G_\bullet) = 0$ for $i > a+1$, $\mathfrak{a}H_{a+1}(G_\bullet) = 0$, and

(1.6.1) $$\mathrm{Im}(\partial_i^{G'}) \cap \mathfrak{a}^s G'_{i-1} \subset \mathfrak{a}^{s-s_i}\mathrm{Im}(\partial_i^{G'})$$ *for all s, i with $s_i \leq s \leq l$, $a+2 \leq i \leq m$.*

If there is a chain map $\lambda'_\bullet : L'_\bullet \to G'_\bullet$ *and* $l \geq n+1+\sum_{j=a+2}^{m} s_j$, *then there is a chain map* $\lambda_\bullet : L_\bullet \to G_\bullet$ *such that*

(i) $\lambda_\bullet|_{L'_\bullet} = \iota_\bullet \circ \lambda'_\bullet$,

(ii) $\mathrm{con}(\lambda'_\bullet)_\bullet$ *is a quasi-direct summand of* $\mathrm{con}(\lambda_\bullet)_\bullet$ *up to* $(\mathfrak{a}^n, m)$,

where $\iota_\bullet : G'_\bullet \to G_\bullet$ *denote the injection and we understand* $\sum_{j=a+2}^{m} s_j = 0$ *in case* $m = a+1$.

Proof. We begin by giving a chain map $\lambda_\bullet : L_\bullet \to G_\bullet$ satisfying (i). By hypotheses, there are a complex $L''_\bullet$ (resp. $G''_\bullet$) of finitely generated graded free modules over A and a chain map $\mu_\bullet : L''_\bullet \to L'[-1]_\bullet$ (resp. $\nu_\bullet : G''_\bullet \to G'[-1]_\bullet$) satisfying $\mathrm{Im}(\mu_i) \subset \mathfrak{a}^l L'_{i-1}$ (resp. $\mathrm{Im}(\nu_i) \subset \mathfrak{a}^l G'_{i-1}$) for all $i \leq m$ (resp. $i \leq m+1$) such that $L_\bullet = \mathrm{con}(\mu_\bullet)_\bullet$ (resp. $G_\bullet = \mathrm{con}(\nu_\bullet)_\bullet$). We have $L_i = L'_i \oplus L''_i$ and $G_i = G'_i \oplus G''_i$ for all $i \in \mathbf{Z}$. Suppose that $l \geq n+1+\sum_{j=a+2}^{m} s_j$ and that there is a chain map $\lambda'_\bullet : L'_\bullet \to G'_\bullet$. We construct a chain map $\lambda_\bullet : L_\bullet \to G_\bullet$ extending $\lambda'_\bullet$, such that

$$\mathrm{pr}_i \circ \lambda_i(L''_i) \subset \mathfrak{a}^{\,l-\sum_{j=a+2}^{i} s_j} G'_i \tag{1.6.2}$$

for $a+1 \leq i \leq m$, where $\mathrm{pr}_i : G_i \to G'_i$ denotes the natural projection and $\sum_{j=a+2}^{i} s_j = 0$ if $i = a+1$. First, let $\lambda_i = 0$ for $i < a+1$. For $a+1 \leq i \leq m$, we will define λ_i inductively as follows. Let $\lambda_{a+1} : L_{a+1} \to G_{a+1}$ be the homomorphism satisfying $\lambda_{a+1}|_{L'_{a+1}} = \begin{pmatrix} \lambda'_{a+1} \\ 0 \end{pmatrix}$ and $\lambda_{a+1}|_{L''_{a+1}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, namely, let

$$\lambda_{a+1} := \begin{pmatrix} \lambda'_{a+1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$\lambda_{i-1} \circ \partial_i^L = \partial_i^G \circ \lambda_i \tag{1.6.3}$$

for all $i \leq a+1$ and the condition (1.6.2) also holds for $i = a+1$. Let i_0 be an integer with $a+1 \leq i_0 < m$ and suppose that we have already defined λ_i for all $i \leq i_0$ satisfying (1.6.2) and (1.6.3). We consider the case $i = i_0+1$. Since $L'_\bullet$ is a quasi-direct summand of $L_\bullet$ up to $(\mathfrak{a}^l, m)$, the set $\partial_{i_0+1}^L(L''_{i_0+1})$ is contained in $\mathfrak{a}^l L'_{i_0} + L''_{i_0}$. Consequently,

$$\begin{aligned}\lambda_{i_0} \circ \partial_{i_0+1}^L(L''_{i_0+1}) &\subset \mathfrak{a}^l \lambda_{a+1}(L'_{a+1}) + \lambda_{a+1}(L''_{a+1}) = \mathfrak{a}^l \lambda_{a+1}(L'_{a+1}) \\ &\subset \mathfrak{a}^l G'_{a+1} \subset \mathfrak{a}\, G_{a+1}\end{aligned}$$

if $i_0 = a+1$, and

$$\begin{aligned}\lambda_{i_0} \circ \partial_{i_0+1}^L(L''_{i_0+1}) &\subset \mathfrak{a}^l \lambda_{i_0}(L'_{i_0}) + \lambda_{i_0}(L''_{i_0}) \\ &\subset \mathfrak{a}^l G_{i_0} + \mathfrak{a}^{\,l-\sum_{j=a+2}^{i_0} s_j}\, G'_{i_0} + G''_{i_0} \\ &\subset \mathfrak{a}^{\,l-\sum_{j=a+2}^{i_0} s_j}\, G'_{i_0} + G''_{i_0}\end{aligned}$$

otherwise. In particular, $\lambda_{a+1} \circ \partial_{a+2}^L(L''_{a+2}) \subset \mathrm{Im}(\partial_{a+2}^G)$ by the hypothesis $\mathfrak{a}\, H_{a+1}(G_\bullet) = 0$. On the other hand, since $\partial_{i_0}^G \circ \lambda_{i_0} \circ \partial_{i_0+1}^L(L''_{i_0+1}) = \lambda_{i_0-1} \circ \partial_{i_0}^L \circ \partial_{i_0+1}^L(L''_{i_0+1}) = 0$ and

$H_i(G_\bullet)=0$ for all $i>a+1$ by our hypotheses, we see $\lambda_{i_0}\circ\partial^L_{i_0+1}(L''_{i_0+1})\subset \mathrm{Im}(\partial^G_{i_0+1})$ also for $i_0>a+1$. Besides,

$$(1.6.4)\qquad \nu_{i_0+1}(G''_{i_0+1})\subset \mathfrak{a}^{\,l}G'_{i_0}.$$

Hence,

$$\begin{aligned}\lambda_{i_0}\circ\partial^L_{i_0+1}(L''_{i_0+1})&\subset \mathrm{Im}(\partial^G_{i_0+1})\cap(\mathfrak{a}^{\,l-\sum_{j=a+2}^{i_0}s_j}G'_{i_0}+G''_{i_0})\\ &\subset \mathrm{Im}(\partial^{G'}_{i_0+1})\cap\mathfrak{a}^{\,l-\sum_{j=a+2}^{i_0}s_j}G'_{i_0}+\mathrm{Im}\begin{pmatrix}(-1)^{i_0+1}\nu_{i_0+1}\\ \partial^{G''}_{i_0+1}\end{pmatrix}\\ &\subset \mathfrak{a}^{\,l-\sum_{j=a+2}^{i_0+1}s_j}\mathrm{Im}(\partial^{G'}_{i_0+1})+\mathrm{Im}\begin{pmatrix}(-1)^{i_0+1}\nu_{i_0+1}\\ \partial^{G''}_{i_0+1}\end{pmatrix}\end{aligned}$$

by (1.6.1) and (1.6.4). There is therefore a homomorphism $\lambda''_{i_0+1}:L''_{i_0+1}\to G_{i_0+1}$ satisfying $\lambda_{i_0}\circ\partial^L_{i_0+1}|_{L''_{i_0+1}}=\partial^G_{i_0+1}\circ\lambda''_{i_0+1}$ and $\mathrm{pr}_{i_0+1}\circ\lambda''_{i_0+1}(L''_{i_0+1})\subset\mathfrak{a}^{\,l-\sum_{j=a+2}^{i_0+1}s_j}G'_{i_0+1}$. Let $\lambda_{i_0+1}:L_{i_0+1}\to G_{i_0+1}$ be the homomorphism such that $\lambda_{i_0+1}|_{L'_{i_0+1}}=\binom{\lambda'_{i_0+1}}{0}$ and $\lambda_{i_0+1}|_{L''_{i_0+1}}=\lambda''_{i_0+1}$. Then λ_{i_0+1} satisfies (1.6.2) and (1.6.3) with $i=i_0+1$. Thus we can obtain $\lambda_{a+1},\dots,\lambda_m$. For $i>m$, the homomorphism λ_i can be defined in the same manner as above, with the requirement (1.6.2) being forgot.

It remains to show that $\lambda_\bullet$ satisfies (ii). Recall that

$$\partial^L_i=\begin{pmatrix}\partial^{L'}_i & (-1)^i\mu_i\\ 0 & \partial^{L''}_i\end{pmatrix}\quad\text{and}\quad \partial^G_i=\begin{pmatrix}\partial^{G'}_i & (-1)^i\nu_i\\ 0 & \partial^{G''}_i\end{pmatrix}.$$

Denoting $\mathrm{con}(\lambda_\bullet)_\bullet$ (resp. $\mathrm{con}(\lambda'_\bullet)_\bullet$) by $C_\bullet$ (resp. $C'_\bullet$), we have $C_i=L_i\oplus G_{i+1}\cong C'_i\oplus C''_i$ with $C'_i=L'_i\oplus G'_{i+1}$, $C''_i=L''_i\oplus G''_{i+1}$, and

$$\partial^C_i=\begin{pmatrix}\partial^{C'}_i & \eta_i\\ 0 & \partial''_i\end{pmatrix}\quad\text{with}\quad \eta_i=\begin{pmatrix}\mu_i & 0\\ \mathrm{pr}_i\circ\lambda_i|_{L''_i} & \nu_{i+1}\end{pmatrix}\quad\text{up to sign,}$$

where $\partial''_i:C''_i\to C''_{i-1}$ is a homomorphism for each i. Since $\mu_i(L''_i)\subset\mathfrak{a}^{\,l}L'_{i-1}$, $\mathrm{pr}_i\circ\lambda_i(L''_i)\subset\mathfrak{a}^{\,l-\sum_{j=a+2}^{i}s_j}G'_i$, and $\nu_{i+1}(G''_{i+1})\subset\mathfrak{a}^{\,l}G'_i$ for all $i\le m$ by what we have seen, and since $l\ge l-\sum_{j=a+2}^{i}s_j\ge n$ $(a+1\le i\le m)$, we find that $\mathrm{Im}(\eta_i)\subset\mathfrak{a}^{\,n}C'_{i-1}$ for all $i\le m$. This shows (ii) by Remark 1.2. □

Lemma 1.7. *Let f be a homogeneous element of A which is A-regular , $d:=\dim(A)$, $\bar A:=A/(f)$, a an integer, and V a finitely generated graded module over A. Let further $Q_\bullet$ and $G_\bullet$ (resp. $\tilde Q_\bullet$* and *$\tilde G_\bullet$) be complexes of finitely generated graded free modules over A (resp. $\bar A$) such that $H_i(Q_\bullet)=0$, $H_i(\tilde Q_\bullet)=0$ for $i\ge a$, $H_i(G_\bullet)=0$, $H_i(\tilde G_\bullet)=0$ for $i>a+1$, and $G_i=0$, $\tilde G_i=0$ for $i<a+1$. Suppose there are chain maps $\alpha_\bullet:Q_\bullet\to\tilde Q_\bullet$, $\beta_\bullet:G_\bullet\to\tilde G_\bullet$, $\lambda_\bullet:Q_\bullet\to G_\bullet$, and $\tilde\lambda_\bullet:\tilde Q_\bullet\to\tilde G_\bullet$ over A satisfying $\beta_\bullet\circ\lambda_\bullet=\tilde\lambda_\bullet\circ\alpha_\bullet$. Denote the mapping cone of $\lambda_\bullet$* (resp. $\tilde\lambda_\bullet$) *by $D_\bullet$ (resp. $\tilde D_\bullet$) and*

$\gamma_\bullet : D_\bullet \to \tilde{D}_\bullet$ the natural chain map induced from $\alpha_\bullet$ and $\beta_\bullet$. Suppose that the canonical homomorphism

$$H_{a+1}(\beta_\bullet) : H_{a+1}(G_\bullet) \to H_{a+1}(\tilde{G}_\bullet)$$

induced from $\beta_\bullet$ is an isomorphism with $H_{a+1}(G_\bullet) \cong H_{a+1}(\tilde{G}_\bullet) \cong V$, *and the canonical homomorphism*

$$H^i_{\mathfrak{m}}(\alpha_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^Q_1)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{Q}}_1))$$

induced from α_0 is an isomorphism for all $0 \le i < d-1$ with $H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^Q_1)) \cong H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{Q}}_1)) = 0$, and that $\dim(V) - d + 2 < a < 0$. Then γ_0 gives rise to the canonical isomorphism

$$H^i_{\mathfrak{m}}(\gamma_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^D_1)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{D}}_1))$$

for all $0 \le i < d-1$ with $H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^D_1)) \cong H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{D}}_1)) = 0$.

Proof. We have a commutative diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G[1]_\bullet & \longrightarrow & D_\bullet & \longrightarrow & Q_\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta[1]_\bullet} & & \downarrow{\scriptstyle \gamma_\bullet} & & {\scriptstyle \alpha_\bullet}\downarrow & & \\ 0 & \longrightarrow & \tilde{G}[1]_\bullet & \longrightarrow & \tilde{D}_\bullet & \longrightarrow & \tilde{Q}_\bullet & \longrightarrow & 0 \end{array} \tag{1.7.1}$$

with exact rows. This induces a commutative diagram of graded modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Coker}(\partial^G_2) & \longrightarrow & \mathrm{Coker}(\partial^D_1) & \longrightarrow & \mathrm{Coker}(\partial^Q_1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{\beta}_1} & & \downarrow{\scriptstyle \bar{\gamma}_0} & & \downarrow{\scriptstyle \bar{\alpha}_0} & & \\ 0 & \longrightarrow & \mathrm{Coker}(\partial^{\tilde{G}}_2) & \longrightarrow & \mathrm{Coker}(\partial^{\tilde{D}}_1) & \longrightarrow & \mathrm{Coker}(\partial^{\tilde{Q}}_1) & \longrightarrow & 0 \end{array}, \tag{1.7.2}$$

where $\bar{\beta}_1$, $\bar{\gamma}_0$ and $\bar{\alpha}_0$ are the natural homomorphisms induced from β_1, γ_0 and α_0, respectively. Since $G[1]_\bullet$, $D_\bullet$, $Q_\bullet$, $\tilde{G}[1]_\bullet$, $\tilde{D}_\bullet$ and $\tilde{Q}_\bullet$ are exact at G_{i+1}, D_i, Q_i, $\tilde{G}_{i+1}$, $\tilde{D}_i$, and $\tilde{Q}_i$, respectively, for all $i \ge 0$ by hypotheses, the rows of (1.7.2) are also exact. Besides, $\mathrm{Coker}(\partial^G_2) \cong \mathrm{Im}(\partial^G_1)$ and $\mathrm{Coker}(\partial^{\tilde{G}}_2) \cong \mathrm{Im}(\partial^{\tilde{G}}_1)$. On the other hand, there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Coker}(\partial^G_2) & \longrightarrow & G_0 & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \bar{\beta}_1} & & \downarrow{\scriptstyle \beta_0} & & \\ 0 & \longrightarrow & \mathrm{Coker}(\partial^{\tilde{G}}_2) & \longrightarrow & \tilde{G}_0 & \longrightarrow & \dots \end{array}$$

$$\begin{array}{ccccccccc} \dots & \longrightarrow & G_{a+2} & \longrightarrow & G_{a+1} & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_{a+2}} & & \downarrow{\scriptstyle \beta_{a+1}} & & \| & & \\ \dots & \longrightarrow & \tilde{G}_{a+2} & \longrightarrow & \tilde{G}_{a+1} & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

where the rows are exact again by our hypotheses. Since $H^i_{\mathfrak{m}}(A)$ and $H^i_{\mathfrak{m}}(\bar{A})$ vanish for all $i < d-1$, the above sequences yield a commutative diagram

$$\begin{array}{ccc} H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^G_2)) & \xrightarrow{\sim} & H^{i+a}_{\mathfrak{m}}(V) \\ \downarrow{\scriptstyle H^i_{\mathfrak{m}}(\beta_1)} & & \| \\ H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{G}}_2)) & \xrightarrow{\sim} & H^{i+a}_{\mathfrak{m}}(V) \end{array}$$

for each $i < d-1$. Hence, $H^i_{\mathfrak{m}}(\beta_1)$ $(0 \le i < d-1)$ are isomorphisms. In addition,

$$H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^G_2)) \cong H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{G}}_2)) \cong H^{d-2+a}_{\mathfrak{m}}(V) = 0,$$

since $\dim(V) < d-2+a$. Now take the commutative diagram of local cohomologies whose rows are the long exact sequences arising from the rows of (1.7.2). Then, with the use of five lemma, we find by hypotheses and by what we have seen that our assertion holds. □

Lemma 1.8. *With the same notation as in the preceding lemma, suppose that the canonical homomorphism $H_i(\alpha_\bullet) : H_i(Q_\bullet) \to H_i(\tilde{Q}_\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$. Then the canonical homomorphism $H_i(\gamma_\bullet) : H_i(D_\bullet) \to H_i(\tilde{D}_\bullet)$ is also an isomorphism for all $i \in \mathbf{Z}$.*

Proof. Our assertion follows from the commutative diagram whose rows are the long exact sequences of homologies arising from the rows of (1.7.1). □

To proceed further, we need the following lemma from [1].

Lemma 1.9. *Let a_0 and a be integers with $a \ge a_0$, and let $F_\bullet$ be a minimal complex of finitely generated graded free modules over A such that $F_i = 0$ for $i < a_0$ and $H_i(F_\bullet) = 0$ for $i > a$. Let further*

$$\cdots \to P_2 \to P_1 \to P_0 \to H_a(F_\bullet) \to 0$$

be a minimal free resolution of $H_a(F_\bullet)$ over A and $G_\bullet := P[-a-1]_\bullet$. Then there exists a minimal complex $L_\bullet$ of finitely generated graded free modules over A and a chain map $\mu_\bullet : L_\bullet \to G_\bullet$ such that $F_\bullet$ is the minimal part of $\mathrm{con}(\mu_\bullet)_\bullet$, *where $L_i = 0$ for $i < a_0$ and $H_i(L_\bullet) = 0$ for $i \ge a$.*

Proof. The proof of [1, (1.5)] works well. □

Lemma 1.10. *Let $a_0 < 0$ be an integer, $d := \dim(A)$, and $F_\bullet$ a minimal complex of finitely generated graded free modules over A such that $F_i = 0$ for $i < a_0$,* $\dim(H_i(F_\bullet))$ *$< d-2+i$ for $a_0 \le i \le -1$, and $H_i(F_\bullet) = 0$ for $i \ge 0$. Let further $\mathfrak{a}$ be a homogeneous ideal in A annihilating all $H_i(F_\bullet)$ $(a_0 \le i \le -1)$ and $n \ge 0$ an integer. Then, there is a positive integer n_0 such that, for an arbitrary homogeneous A-regular element f of $\mathfrak{a}^{n_0}$, there are a complex $D_\bullet$ of finitely generated graded free modules over A and a complex $\tilde{D}_\bullet$ of finitely generated graded free modules over $\bar{A} := A/(f)$, satisfying the following conditions:*

(i) $F_\bullet$ is the minimal part of $D_\bullet$.
(ii) *$\bar{D}_\bullet := D_\bullet / f D_\bullet$ is a quasi-direct summand of $\tilde{D}_\bullet$ up to $(\mathfrak{a}^n, 0)$.*
(iii) *$D_i = 0$ and $\tilde{D}_i = 0$ for $i < a_0$.*
(iv) *The canonical homomorphism $H_i(\nu_\bullet) : H_i(D_\bullet) \to H_i(\tilde{D}_\bullet)$, induced from the chain map $\nu_\bullet : D_\bullet \to \tilde{D}_\bullet$ over A obtained by composing the natural surjection $D_\bullet \twoheadrightarrow \bar{D}_\bullet$ and the injection $\bar{D}_\bullet \hookrightarrow \tilde{D}_\bullet$, is an isomorphism for all $i \in \mathbf{Z}$.*

(v) *The canonical homomorphism*

$$H^i_{\mathfrak{m}}(v_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^D)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^{\tilde{D}}))$$

induced from v_0 *is an isomorphism for all* $0 \le i < d-1$ *with*

$$H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^D)) \cong H^{d-2}_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^{\tilde{D}})) = 0.$$

Proof. We prove our assertion by induction on $a := \max(\{i | H_i(F_\bullet) \neq 0\} \cup \{a_0 - 1\})$. Note that $a_0 - 1 \le a < 0$.

Consider first the case $a = a_0 - 1$. In this case, we see $H_i(F_\bullet) = 0$ for all $i \ge a_0$, so that $H_i(F_\bullet) = 0$ for all $i \in \mathbf{Z}$ by hypotheses. Since $F_\bullet$ is an exact minimal complex bounded below, this means that $F_i = 0$ for all $i \in \mathbf{Z}$. It is therefore enough to set $D_\bullet := F_\bullet$, $\tilde{D}_\bullet := F_\bullet / f F_\bullet$.

Assume that $a \ge a_0$ and that our assertion is true for smaller values of a. Let $V := H_a(F_\bullet)$,

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0$$

a minimal free resolution of V over A, $P_i := 0$ for $i < 0$, and $G_\bullet := P[-a-1]_\bullet$. By Lemma 1.9, $F_\bullet$ is the minimal part of the mapping cone $C_\bullet := \mathrm{con}(\mu_\bullet)_\bullet$ of a chain map $\mu_\bullet : L_\bullet \to G_\bullet$, where $L_\bullet$ is a minimal complex of finitely generated graded free modules over A satisfying $L_i = 0$ for $i < a_0$ and $H_i(L_\bullet) = 0$ for $i \ge a$. We have a short exact sequence

$$(1.10.1) \quad 0 \to G[1]_\bullet \to C_\bullet \to L_\bullet \to 0$$

and an equality $C_\bullet = F_\bullet \oplus \mathrm{se}(C_\bullet)_\bullet$. Since $H_i(L_\bullet) \cong H_i(C_\bullet) \cong H_i(F_\bullet)$ for $i < a$ by (1.10.1) and since $H_i(L_\bullet) = 0$ for $i \ge a$, $L_\bullet$ satisfies $\dim(H_i(L_\bullet)) < d - 2 + i$ for $a_0 \le i \le -1$ by hypotheses. Besides, $\max(\{i | H_i(L_\bullet) \neq 0\} \cup \{a_0 - 1\}) < a$. By Artin–Rees lemma, there is an integer $s_i \ge 0$ for each $a + 2 \le i \le 0$ such that

$$(1.10.2) \quad \mathrm{Im}(\partial_i^G) \cap \mathfrak{a}^s G_{i-1} \subset \mathfrak{a}^{s-s_i} \mathrm{Im}(\partial_i^G) \quad \text{for all } s \ge s_i.$$

Put $l_1 := n + 1 + \sum_{j=a+2}^{0} s_j$, where $\sum_{j=a+2}^{0} s_j = 0$ in case $a + 2 = 1$. By the induction hypothesis, there is an integer $l_0 > 0$ such that for an arbitrary homogeneous A-regular element f of $\mathfrak{a}^{l_0}$, there are a complex $Q_\bullet$ of finitely generated graded free modules over A, and a complex $\tilde{Q}_\bullet$ of finitely generated graded free modules over $\bar{A} := A/(f)$ such that

(i′) $L_\bullet$ is the minimal part of $Q_\bullet$,
(ii′) $\bar{Q}_\bullet := Q_\bullet / f Q_\bullet$ is a quasi-direct summand of $\tilde{Q}_\bullet$ up to $(\mathfrak{a}^{l_1}, 0)$,
(iii′) $Q_i = 0$ and $\tilde{Q}_i = 0$ for $i < a_0$,
(iv′) the canonical homomorphism $H_i(\alpha_\bullet) : H_i(Q_\bullet) \to H_i(\tilde{Q}_\bullet)$, induced from the chain map $\alpha_\bullet : Q_\bullet \to \tilde{Q}_\bullet$ over A obtained by composing the natural surjection $Q_\bullet \twoheadrightarrow \bar{Q}_\bullet$ and the injection $\bar{Q}_\bullet \hookrightarrow \tilde{Q}_\bullet$, is an isomorphism for all $i \in \mathbf{Z}$,
(v′) the canonical homomorphism

$$H^i_{\mathfrak{m}}(\alpha_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^Q)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^{\tilde{Q}}))$$

induced from α_0 is an isomorphism for all $0 \le i < d-1$ with

$$H_{\mathfrak{m}}^{d-2}(\mathrm{Coker}(\partial_1^{Q})) \cong H_{\mathfrak{m}}^{d-2}(\mathrm{Coker}(\partial_1^{\tilde{Q}})) = 0.$$

Apply Lemma 1.5 to $P_\bullet$. Then there exists an integer $l_2 > 0$ such that, for an arbitrary homogeneous A-regular element f of $\mathfrak{a}^{l_2}$, there is a complex $\tilde{P}_\bullet$ of finitely generated graded free modules over $\bar{A} := A/(f)$ with $\tilde{P}_i = 0$ for $i < 0$ giving a free resolution

$$\cdots \to \tilde{P}_1 \to \tilde{P}_0 \to V \to 0$$

of V over $\bar{A}$, which satisfies the following conditions:

(vi′) $\bar{P}_\bullet := P_\bullet / fP_\bullet$ is a quasi-direct summand of $\tilde{P}_\bullet$ up to $(\mathfrak{a}^{l_1}, -a)$.

(vii′) The canonical homomorphism $H_i(\tilde{\beta}_\bullet) : H_i(P_\bullet) \to H_i(\tilde{P}_\bullet)$, induced from the chain map $\tilde{\beta}_\bullet : P_\bullet \to \tilde{P}_\bullet$ over A obtained by composing the natural surjection $P_\bullet \twoheadrightarrow \bar{P}_\bullet$ and the injection $\bar{P}_\bullet \hookrightarrow \tilde{P}_\bullet$, is an isomorphism for all $i \in \mathbf{Z}$.

Put $n_0 = \max(l_0, l_1, l_2)$. Let f be a homogeneous A-regular element of $\mathfrak{a}^{n_0}$ and $\bar{A} := A/(f)$. Since $f \in \mathfrak{a}^{n_0} \subset \mathfrak{a}^{l_i}$ for $i = 0, 2$, there are $Q_\bullet$, $\tilde{Q}_\bullet$ and $\tilde{P}_\bullet$ satisfying the conditions stated above. Let $\bar{G}_\bullet := G_\bullet / fG_\bullet$, $\tilde{G}_\bullet := \tilde{P}[-a-1]_\bullet$, $\beta_\bullet := \tilde{\beta}[-a-1]_\bullet : G_\bullet \to \tilde{G}_\bullet$. Then $\bar{G}_\bullet$ is a quasi-direct summand of $\tilde{G}_\bullet$ up to $(\mathfrak{a}^{l_1}, 1)$. Besides, the canonical homomorphism $H_{a+1}(\beta_\bullet) : H_{a+1}(G_\bullet) \to H_{a+1}(\tilde{G}_\bullet)$ induced from $\beta_\bullet$ is an isomorphism. By (1.10.2) and Lemma 1.3, the complex $\bar{G}_\bullet$ satisfies

$$\mathrm{Im}(\partial_i^{\bar{G}}) \cap \mathfrak{a}^s \bar{G}_{i-1} \subset \mathfrak{a}^{s-s_i} \mathrm{Im}(\partial_i^{\bar{G}}) \quad \text{for all } s, i \text{ with } s_i \le s \le n_0,\ a+2 \le i \le 0.$$

Using the equality $Q_\bullet = L_\bullet \oplus \mathrm{se}(Q_\bullet)_\bullet$, we define $\lambda_\bullet : Q_\bullet \to G_\bullet$ to be the chain map such that $\lambda_\bullet|_{L_\bullet} = \mu_\bullet$ and $\lambda_\bullet|_{\mathrm{se}(Q_\bullet)_\bullet} = 0$. Since $Q_\bullet = L_\bullet \oplus \mathrm{se}(Q_\bullet)_\bullet$ and $C_\bullet = F_\bullet \oplus \mathrm{se}(C_\bullet)_\bullet$, we find $\mathrm{con}(\lambda_\bullet)_\bullet = \mathrm{con}(\mu_\bullet)_\bullet \oplus \mathrm{se}(Q_\bullet)_\bullet = C_\bullet \oplus \mathrm{se}(Q_\bullet)_\bullet = F_\bullet \oplus \mathrm{se}(C_\bullet)_\bullet \oplus \mathrm{se}(Q_\bullet)_\bullet$. This implies that $F_\bullet$ is the minimal part of $D_\bullet := \mathrm{con}(\lambda_\bullet)_\bullet$, namely, this $D_\bullet$ satisfies (i). Apply Lemma 1.6 to the complexes $\bar{Q}_\bullet$, $\tilde{Q}_\bullet$, $\bar{G}_\bullet$, $\tilde{G}_\bullet$ over $\bar{A}$ and to the natural chain map $\bar{\lambda}_\bullet : \bar{Q}_\bullet \to \bar{G}_\bullet$ induced from $\lambda_\bullet$, with $m = 0$, $l = l_1$. Then there is a chain map $\tilde{\lambda}_\bullet : \tilde{Q}_\bullet \to \tilde{G}_\bullet$ such that $\tilde{\lambda}_\bullet|_{\bar{Q}_\bullet} = \iota_\bullet \circ \bar{\lambda}_\bullet$ and $\mathrm{con}(\bar{\lambda}_\bullet)_\bullet$ is a quasi-direct summand of $\mathrm{con}(\tilde{\lambda}_\bullet)_\bullet$ up to $(\mathfrak{a}^n, 0)$, where $\iota_\bullet : \bar{G}_\bullet \to \tilde{G}_\bullet$ denotes the injection. Let $\bar{D}_\bullet := D_\bullet / fD_\bullet$. Since $\bar{D}_\bullet$ coincides with $\mathrm{con}(\bar{\lambda}_\bullet)_\bullet$, it satisfies (ii) also. The condition (iii) is trivial by our construction. Since $V = H_a(F_0) \cong H_{a+1}(G_\bullet) \cong H_{a+1}(\tilde{G}_\bullet)$, we see $\dim(V) - d + 2 < a < 0$ by hypotheses. Properties (iv) and (v), therefore, follow from (iv′) and (v′), and Lemmas 1.7 and 1.8. □

Theorem 1.11. *Let a_0, p be integers with $a_0 < 0$, $2 \le p \le \dim(A)$ and $F_\bullet$ a minimal complex of graded free modules over A such that $F_i = 0$ for $i < a_0$, $\dim(H_i(F_\bullet)) \le \dim(A) - p + i$ for $a_0 \le i \le -1$, and $H_i(F_\bullet) = 0$ for $i \ge 0$. Let further $\mathfrak{a}$ be a homogeneous ideal in A of grade larger than or equal to $p-2$ annihilating all $H_i(F_\bullet)$ $(a_0 \le i \le -1)$. Then, there are a homogeneous A-regular sequence $f_1, \ldots, f_{p-2}$ with $f_i \in \mathfrak{a}$ for all $1 \le i \le p-2$, a minimal complex $\tilde{F}_\bullet$ of finitely generated graded free modules over $A/(f_1, \ldots, f_{p-2})$, and a chain map $\tau_\bullet : F_\bullet \to \tilde{F}_\bullet$ over A satisfying* *the following conditions*:

(i) $\tilde{F}_i = 0$ *for* $i < a_0$.

(ii) *The canonical homomorphism*

$$H^i_{\mathfrak{m}}(\tau_0): H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^F_1)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{F}}_1))$$

induced from τ_0 *is an isomorphism for all* $0 \le i < \dim(A) - p + 2$.

(iii) *The canonical homomorphism* $H_i(\tau_\bullet): H_i(F_\bullet) \to H_i(\tilde{F}_\bullet)$ *induced from* $\tau_\bullet$ *is an isomorphism for all* $i \in \mathbf{Z}$.

Moreover, we can choose $f_1, \ldots, f_{p-2}$ *so that* $\mathrm{Proj}(A/(f_1, \ldots, f_{p-2}))$ *is smooth in the outside of the union of* $\mathrm{Proj}(A/\mathfrak{a})$ *and the singularity of* $\mathrm{Proj}(A)$.

Proof. Let $d := \dim(A)$. When $p = 2$, we have nothing to do. Just set $\tilde{F}_\bullet = F_\bullet$ and define $\tau_\bullet$ to be the identity chain map. Suppose that $2 < p \le d$ and that our assertion is true for smaller values of p. Then by Lemma 1.10, there is a positive integer n_0, a homogeneous A-regular element f_1 of $\mathfrak{a}^{n_0}$, a complex $D_\bullet$ of finitely generated graded free modules over A, and a complex $\tilde{D}_\bullet$ of finitely generated graded free modules over $\bar{A} := A/(f_1)$ satisfying conditions (i)–(v) stated there with $n = 0$. Here, we may assume with no loss of generality that f_1 is a sufficiently general element of $[\mathfrak{a}^{n_0}]_l$ with l large enough. By Lemma 1.12 below, therefore, $\mathrm{Proj}(\bar{A})$ is smooth in the outside the union of $\mathrm{Proj}(A/\mathfrak{a})$ and the singularity of $\mathrm{Proj}(A)$. Let $\tilde{F}'_\bullet$ be $\min(\tilde{D}_\bullet)_\bullet$, $\sigma_\bullet: F_\bullet \to D_\bullet$ the inclusion, $\varpi_\bullet: \tilde{D}_\bullet \to \tilde{F}'_\bullet$ the natural projection, and $\tau'_\bullet := \varpi_\bullet \circ v_\bullet \circ \sigma_\bullet$, where $v_\bullet: D_\bullet \to \tilde{D}_\bullet$ denotes the chain map over A mentioned in (iv) of Lemma 1.10. Since $D_\bullet = F_\bullet \bigoplus \mathrm{se}(D_\bullet)_\bullet$ (resp. $\tilde{D}_\bullet = \tilde{F}'_\bullet \oplus \mathrm{se}(\tilde{D}_\bullet)_\bullet$) with $\mathrm{se}(D_\bullet)_\bullet$ (resp. $\mathrm{se}(\tilde{D}_\bullet)_\bullet$) split exact, the canonical homomorphisms $H_i(\sigma_\bullet): H_i(F_\bullet) \to H_i(D_\bullet)$ and $H_i(\varpi_\bullet): H_i(\tilde{D}_\bullet) \to H_i(\tilde{F}'_\bullet)$ are isomorphisms for all $i \in \mathbf{Z}$. Moreover, since $\mathrm{Coker}(\partial_1^{\mathrm{se}(D_\bullet)})$ (resp. $\mathrm{Coker}(\partial_1^{\mathrm{se}(\tilde{D}_\bullet)})$) is free over A (resp. $\bar{A}$), the canonical homomorphism $H^i_{\mathfrak{m}}(\sigma_0): H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^F_1)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^D_1))$ (resp. $H^i_{\mathfrak{m}}(\varpi_0): H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{D}}_1)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{F}'}_1))$) is an isomorphism for all $0 \le i < d$ (resp. $0 \le i < d-1$). Hence by (i)–(v) of Lemma 1.10,

(i′) $\tilde{F}'_i = 0$ for $i < a_0$,

(ii′) the canonical homomorphism

$$H^i_{\mathfrak{m}}(\tau'_0): H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^F_1)) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial^{\tilde{F}'}_1))$$

induced from τ'_0 is an isomorphism for all $0 \le i < d - 1$,

(iii′) the canonical homomorphism $H_i(\tau'_\bullet): H_i(F_\bullet) \to H_i(\tilde{F}'_\bullet)$ induced from $\tau'_\bullet$ is an isomorphism for all $i \in \mathbf{Z}$.

Moreover, $H_i(\tilde{F}'_\bullet) = 0$ for $i \ge 0$ and $\dim(H_i(\tilde{F}'_\bullet)) \le (d-1) - (p-1) + i$ for $a_0 \le i \le -1$ by (iii′) and our hypotheses. Let $\bar{\mathfrak{a}} := \mathfrak{a}^{n_0}/(f_1)$. Then by the induction hypothesis, there are a homogeneous $\bar{A}$-regular sequence $\bar{f}_2, \ldots, \bar{f}_{p-2}$ with $\bar{f}_i \in \bar{\mathfrak{a}}$ for all $2 \le i \le p-2$, a minimal complex $\tilde{F}_\bullet$ of finitely generated graded free modules over $\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{p-2}) = A/(f_1, \ldots, f_{p-2})$, and a chain map $\tau''_\bullet: \tilde{F}'_\bullet \to \tilde{F}_\bullet$ over $\bar{A}$ satisfying the following conditions:

(i$''$) $\tilde{F}_i = 0$ for $i < a_0$.

(ii$''$) The canonical homomorphism

$$H^i_{\mathfrak{m}}(\tau''_0) : H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^{\tilde{F}'})) \to H^i_{\mathfrak{m}}(\mathrm{Coker}(\partial_1^{\tilde{F}}))$$

induced from τ''_0 is an isomorphism for all $0 \leq i < (d-1)-(p-1)+2 = d-p+2$.

(iii$''$) The canonical homomorphism $H_i(\tau''_\bullet) : H_i(\tilde{F}'_\bullet) \to H_i(\tilde{F}_\bullet)$ induced from $\tau''_\bullet$ is an isomorphism for all $i \in \mathbf{Z}$.

Moreover, we can choose $\bar{f}_2, \ldots, \bar{f}_{p-2}$ so that $\mathrm{Proj}(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{p-2}))$ is smooth in the outside of the union of $\mathrm{Proj}(\bar{A}/\bar{\mathfrak{a}}) = \mathrm{Proj}(A/\mathfrak{a})$ and the singularity of $\mathrm{Proj}(\bar{A})$. Let $f_2, \ldots, f_{p-2}$ be the homogeneous elements of A such that $f_i + (f_1) = \bar{f}_i$ $(2 \leq i \leq p-2)$ and let $\tau_\bullet := \tau''_\bullet \circ \tau'_\bullet$. Then $\tilde{F}_\bullet$ and $\tau_\bullet$ satisfies (i)–(iii). The last assertion is also true by what we have seen, since the singularity of $\mathrm{Proj}(\bar{A})$ is contained in the union of $\mathrm{Proj}(A/\mathfrak{a})$ and the singularity of $\mathrm{Proj}(A)$. □

Lemma 1.12. *Let $\mathfrak{a}$ be a homogeneous ideal in A and $l > 0$ an integer such that the subideal $\bigoplus_{j \geq l} [\mathfrak{a}]_j$ of $\mathfrak{a}$ is generated over A by $[\mathfrak{a}]_l$. Let Z be the singularity of $X := \mathrm{Proj}(A)$ and Z' the subscheme of X defined by $\mathfrak{a}$. Then for every sufficiently general homogeneous element $f \in [\mathfrak{a}]_{l+1}$, the scheme $\mathrm{Proj}(A/(f))$ is smooth in the outside of $Z \cup Z'$.*

Proof. The base locus of the linear system on X generated by the elements of $[\mathfrak{a}]_l$ coincides with Z'. Hence a general member of the linear system on X generated by the elements of $[\mathfrak{a}]_{l+1} = [\mathfrak{a}]_l[R]_1$ is smooth in the outside of $Z \cup Z'$ (cf. [9]). This proves our assertion. □

2. Existence of homogeneous prime ideals

Borrowing the idea of orientation from [4,5,8], we will say that a coherent sheaf $\mathscr{M}$ of rank v on a projective scheme $X \hookrightarrow \mathrm{Proj}(R)$ is *orientable* on a Zariski open subset U of X, if $\bigwedge^v \mathscr{M}|_U \cong \mathscr{O}_U(n)$ for some integer n.

Lemma 2.1. *Let a_0 be an integer with $a_0 < 0$ and $F_\bullet$ a complex of graded free modules over A such that $F_i = 0$ for $i < a_0$ and $H_i(F_\bullet) = 0$ for $i \geq 0$. Let further M be* $\mathrm{Coker}(\partial_1^F)$ *and $\mathscr{M}$ the sheaf on $X := \mathrm{Proj}(A)$ defined by M. If there is a homogeneous ideal $\mathfrak{a} \subset R$ such that $H_i(F_\bullet)_\mathfrak{p} = 0$ for all $a_0 \leq i \leq -1$ and all homogeneous prime ideals $\mathfrak{p}$ not containing $\mathfrak{a}$, then $\mathscr{M}$ is locally free and orientable on the outside of* $\mathrm{Proj}(A/\mathfrak{a})$.

Proof. Let $Z := \mathrm{Proj}(A/\mathfrak{a})$. Denote by $\mathscr{F}_i$ the sheaf on X defined by the graded free module F_i for each $i \in \mathbf{Z}$. Then the complex

$$0 \to \mathscr{M} \to \mathscr{F}_{-1} \xrightarrow{\partial^F_{-1}} \mathscr{F}_{-2} \xrightarrow{\partial^F_{-2}} \cdots \xrightarrow{\partial^F_{a_0+1}} \mathscr{F}_{a_0} \to 0$$

induced from $F_\bullet$ is exact on the outside of Z, since $H_i(F_\bullet)_\mathfrak{p} = 0$ for all $a_0 \le i \le -1$ and all prime ideals $\mathfrak{p}$ not containing $\mathfrak{a}$. Hence $\mathscr{M}$ is locally free and orientable on the outside of Z. $\square$

Let M be a finitely generated graded module over R with no free direct summand,

$$\cdots \xrightarrow{\partial_{p+1}} F_p \xrightarrow{\partial_p} F_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial'_0} M \to 0$$

a minimal free resolution of M over R, and

$$0 \to F''_{a_0} \xrightarrow{\partial''_{a_0-1}} \cdots \xrightarrow{\partial''_{-2}} F''_{-2} \xrightarrow{\partial''_{-1}} F''_{-1} \xrightarrow{\partial''_0} M^\vee \to 0$$

a minimal free resolution of $M^\vee$ over R, where $a_0 < 0$. Let further $F_i := F_i''^\vee$ and $\partial_i = \partial_i''^\vee$ for $i < 0$, and $\partial_0 := (\partial_0'^\vee \circ \partial_0'')^\vee$. Connecting the former resolution to the dual of the latter with the use of ∂_0, we obtain a complex

$$F_\bullet : \cdots \xrightarrow{\partial_{p+1}} F_p \xrightarrow{\partial_p} F_{p-1} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} \cdots \xrightarrow{\partial_{a_0-1}} F_{a_0} \to 0$$

bounded on both sides (cf. [1,3]) such that $H_i(F_\bullet) = 0$ for $i > 0$ and $H_i(F_\bullet^\vee) = 0$ for $i \le 0$, where $F_i = F_i^{\vee\vee}$ and $\partial_i = \partial_i^{\vee\vee}$ for $i < 0$. We will denote this complex by $\mathrm{cpx}(M)_\bullet$.

Theorem 2.2. *Let p, u be integers with $2 \le p \le r$, $u > p$, $\mathfrak{a}$ a homogeneous ideal in R of height larger than or equal to u, and M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}^i_R(M,R) = 0$ for $1 \le i \le p-1$ such that $M_\mathfrak{p}$ is free for all homogeneous prime ideals $\mathfrak{p} \subset R$ not containing $\mathfrak{a}$. Then, there are homogeneous R-regular sequence $f_1, \ldots, f_{p-2}$ with $f_i \in \mathfrak{a}$ for all $1 \le i \le p-2$, a finitely generated graded module $\tilde{M}$ over $\bar{R} := R/(f_1, \ldots, f_{p-2})$, and a homomorphism $\varphi : M \to \tilde{M}$ over R, satisfying the following conditions*:

(i) *The canonical homomorphism $H^i_\mathfrak{m}(\varphi) : H^i_\mathfrak{m}(M) \to H^i_\mathfrak{m}(\tilde{M})$ induced from φ is an isomorphism for all $0 \le i < r-p+2$ and $H^{r-p+1}_\mathfrak{m}(\tilde{M}) = 0$.*
(ii) *The scheme $X := \mathrm{Proj}(\bar{R})$ is an integral normal scheme which is smooth in outside of its subscheme of codimension $u-p+2$.*
(iii) *The sheaf $\tilde{\mathscr{M}}$ on X that $\tilde{M}$ defines is locally free and orientable on the outside of a subscheme of X of codimension $u-p+2$.*
(iv) *$\tilde{M}$ is torsion-free over the integral domain $\bar{R}$.*

Proof. Let $F_\bullet := \mathrm{cpx}(M)_\bullet$. Since, $H^i(F_\bullet^\vee) = 0$ for $i \le 0$ by our construction and $H^i(F_\bullet^\vee) = \mathrm{Ext}^i_R(M,R) = 0$ for $1 \le i \le p-1$, we see

$$\dim(H_i(F_\bullet)) = \dim(\mathrm{Ext}_R^{p-i}(\mathrm{Coker}(\partial_p^{F\vee}), R)) \le r - p + i \quad \text{for all } i < p.$$

Moreover, $(F_\bullet)_\mathfrak{p}$ is exact, namely, $H_i(F_\bullet)_\mathfrak{p} = 0$ for all $i \in \mathbf{Z}$ and all homogeneous prime ideals $\mathfrak{p}$ not containing $\mathfrak{a}$, by our construction of $F_\bullet$ and the hypothesis that $M_\mathfrak{p}$ is free for such prime ideals. There is therefore an integer $n > 0$ such that $\mathfrak{a}^n H_i(F_\bullet) = 0$ for all $a_0 \le i \le -1$. Let $\mathfrak{a}' := \mathfrak{a}^n$. Then the grade of $\mathfrak{a}'$ is larger than or equal to $u > p$. Recall here that $F_\bullet$ is exact also at F_0 by the torsion-freeness of M (see the proof of [3, Theorem 3]). By Theorem 1.11, there are a homogeneous R-regular sequence $f_1,\dots,f_{p-2}$ with $f_i \in \mathfrak{a}' \subset \mathfrak{a}$ for all $1 \le i \le p-2$, a minimal complex $\tilde{F}_\bullet$ of finitely generated graded free modules over $\bar{R} := R/(f_1,\dots,f_{p-2})$, and a chain map $\tau_\bullet : F_\bullet \to \tilde{F}_\bullet$ over R satisfying the conditions (i), (ii) and (iii) stated in that theorem. Let $\tilde{M} := \mathrm{Coker}(\partial_1^{\tilde{F}})$ and let φ denote the homomorphism from $M = \mathrm{Coker}(\partial_1^F)$ to $\tilde{M}$ obtained from $\tau_\bullet$ or rather from τ_0 in the natural manner. Then the canonical homomorphism $H^i_{\mathfrak{m}}(\varphi) : H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(\tilde{M})$ induced from φ is an isomorphism for all $0 \le i < r-p+2$ by (ii) of Theorem 1.11. Moreover, since $H^{r-p+1}_{\mathfrak{m}}(M) = 0$ by local duality and hypotheses, we see $H^{r-p+1}_{\mathfrak{m}}(\tilde{M}) = 0$. This proves (i). By the last part of Theorem 1.11, we may assume that $X := \mathrm{Proj}(\bar{R})$ is smooth in the outside of $\mathrm{Proj}(R/\mathfrak{a}')$. Since $\mathrm{ht}(\mathfrak{a}') = \mathrm{ht}(\mathfrak{a}) \ge u > p$, this implies that the codimension of the singularity of X is larger than or equal to $\mathrm{ht}(\mathfrak{a}') - p + 2 \ge u - p + 2 \ge 3$ in X. On the other hand, X is a complete intersection, so that it is connected and locally Cohen–Macaulay. Therefore X is normal by Serre's criterion of normality (see e.g. [10, Theorem 39]). Since, a normal scheme is locally irreducible, X must be integral. Hence (ii) holds. Note that, as its consequence, $\bar{R}$ is an integral domain. Let i be an integer, $\bar{\mathfrak{p}}$ a homogeneous prime ideal in $\bar{R}$ not containing $\bar{\mathfrak{a}} := \mathfrak{a}/(f_1,\dots,f_{p-2})$, and $\mathfrak{p} \supset (f_1,\dots,f_{p-2})$ the homogeneous prime ideal in R such that $\mathfrak{p}/(f_1,\dots,f_{p-2}) = \bar{\mathfrak{p}}$. Then, $\mathfrak{p} \not\supset \mathfrak{a}$ and $H_i(F_\bullet)_{\mathfrak{p}} = 0$, so that $H_i(\tilde{F}_\bullet)_{\bar{\mathfrak{p}}} = H_i(F_\bullet)_{\mathfrak{p}} = 0$ since $H_i(\tau_\bullet) : H_i(F_\bullet) \to H_i(\tilde{F}_\bullet)$ is an isomorphism. Hence (iii) holds by Lemma 2.1. Finally, since $H_0(\tilde{F}_\bullet) \cong H_0(F_\bullet) = 0$, we have $\tilde{M} = \mathrm{Im}(\partial_0^{\tilde{F}})$. This implies (iv). $\square$

Lemma 2.3. *Let p, q be integers with $2 \le p \le r$, $1 \le q < p$, M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}^i_R(M,R) = 0$ for $1 \le i \le p-1$, $f_1,\dots,f_q$ homogeneous polynomials of R forming an R-regular sequence, A the factor ring $R/(f_1,\dots,f_q)$, $\tilde{M}$ a finitely generated graded module over A, $\varphi : M \to \tilde{M}$ a homomorphism over R such that $H^i_{\mathfrak{m}}(\varphi) : H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(\tilde{M})$ is an isomorphism for all $i \le r-p$. Suppose there exists a homogeneous ideal $\tilde{I}$ of height $p-q$ in A and a homomorphism $\tilde{\psi} : \tilde{M} \to \tilde{I}(c)$ $(c \in \mathbf{Z})$ such that $H^i_{\mathfrak{m}}(\tilde{\psi}) : H^i_{\mathfrak{m}}(\tilde{M}) \to H^i_{\mathfrak{m}}(\tilde{I}(c))$ is an isomorphism for all $i \le r-p$. Then there is a homogeneous ideal I of height p in R containing $f_1,\dots,f_q$ with $I/(f_1,\dots,f_q) = \tilde{I}$ and a homomorphism $\psi : M \to I(c)$ such that $H^i_{\mathfrak{m}}(\psi) : H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(I(c))$ is an isomorphism for all $i \le r-p$.*

Proof. Let $\pi : R \to A$ be the canonical surjection, $I := \pi^{-1}(\tilde{I})$, and $J := (f_1,\dots,f_q) \subset R$. Since J is a complete intersection, one sees that

(2.3.1) $\quad H^i_{\mathfrak{m}}(J) = 0 \quad$ for all $i \le r-q$

and that J has a free resolution of the form

$$0 \to K_q \to \cdots \to K_1 \to J \to 0.$$

This resolution, together with the hypothesis on the extensions of M, yields $\mathrm{Ext}^1_R(M,J) = 0$. There is, therefore, a homomorphism $\psi : M \to I(c)$ which makes the following diagram commutative

$$\begin{array}{ccccccccc} & & & & M & \xrightarrow{\varphi} & \tilde{M} & & \\ & & & & \Big\downarrow \psi & & \Big\downarrow \tilde{\psi} & & \\ 0 & \longrightarrow & J & \longrightarrow & I(c) & \xrightarrow{\pi} & \tilde{I}(c) & \longrightarrow & 0. \end{array}$$

By (2.3.1), the homomorphism $H^i_{\mathfrak{m}}(\pi) : H^i_{\mathfrak{m}}(I(c)) \to H^i_{\mathfrak{m}}(\tilde{I}(c))$ is an isomorphism for $i < r-q$ and is injective for $i = r-q$. Since $r-p < r-q$, we find that $H^i_{\mathfrak{m}}(\psi)$ is an isomorphism for all $i \leq r-p$ by the above commutative diagram. $\square$

Lemma 2.4. *Let $s \geq 4$ be an integer, X the projective scheme* Proj(A), *Z a subscheme of X of codimension larger than or equal to s. Assume that X is an integral normal scheme which is smooth in the outside of Z. Let $\mathscr{M}$ be a torsion-free coherent sheaf on X of rank $t+1 \geq 2$ which is locally free and orientable on the outside of Z and m an integer such that $\mathscr{M}(m)$ is generated over $\mathscr{O}_X$ by its global sections. Let further n be the integer such that $\bigwedge^{t+1} \mathscr{M}|_{X\setminus Z} \cong \mathscr{O}_{X\setminus Z}(n)$. Then for all integers $m_1, \dots, m_t$ larger than m, there exists a two-codimensional closed subscheme Y of X smooth in the outside of a subscheme of X of codimension not less than* $\min(s,6)$, *whose ideal sheaf $\mathscr{I}_Y$ fits into a Bourbaki sequence*

$$0 \to \bigoplus_{i=1}^{t} \mathscr{O}_X(-m_i) \to \mathscr{M} \to \mathscr{I}_Y(c) \to 0,$$

where $c = n + \sum_{i=1}^t m_i$.

Proof. Denote the Zariski open subset $X \setminus Z$ of X by U. Choose a sufficiently general global section of $\mathscr{M}(m_i)$ for each $1 \leq i \leq t$ and, using them, construct a homomorphism $\varepsilon \colon \bigoplus_{i=1}^t \mathscr{O}_X(-m_i) \to \mathscr{M}$ in the standard manner. Since $\mathscr{M}|_U$ is locally free and orientable, there is, by Bertini's theorem (see [9]), a two-codimensional subscheme Y' of U smooth in the outside of a subscheme Z_1' of U of codimension not less than 6, whose ideal sheaf $\mathscr{I}_{Y'}$ fits into a Bourbaki sequence

$$0 \to \bigoplus_{i=1}^{t} \mathscr{O}_U(-m_i) \xrightarrow{\varepsilon|_U} \mathscr{M}|_U \xrightarrow{\rho'} \mathscr{I}_{Y'}(c) \to 0, \tag{2.4.1}$$

where $c := n + \sum_{i=1}^t m_i$ and ρ' is the homomorphism induced from $\bigwedge^t (\varepsilon|_U)^\vee$. Let $\iota : U \to X$ be the inclusion map and $\rho_1 : \mathscr{O}_X(c) \to \iota_*(\mathscr{O}_U(c))$ the canonical homomorphism. Since $\mathscr{O}_X$ is normal at every point of X by hypothesis, ρ_1 is a bijection. Let $\rho_2 : \mathscr{M} \to \iota_*(\mathscr{M}|_U)$ and $\iota_*(\rho') : \iota_*(M|_U) \to \iota_*(\mathscr{I}_{Y'}(c))$ be the canonical homomorphisms, and $\rho_3 : \iota_*(\mathscr{I}_{Y'}(c)) \to \iota_*(\mathscr{O}_U(c))$ be the inclusion. Put $\rho := \rho_1^{-1} \circ \rho_3 \circ \iota_*(\rho') \circ \rho_2$ and $\mathscr{I} := \mathrm{Im}(\rho)(-c) \subset \mathscr{O}_X$. The ideal sheaf $\mathscr{I}$ satisfies $\mathscr{I}|_U = \mathscr{I}_{Y'}$ by definition, so that there is a two-codimensional closed subscheme Y of X such that $\mathscr{I} = \mathscr{I}_Y$. Note that $Y \cap U = Y'$ and that Y is smooth in the outside of $Z \cup \overline{Z_1'}$. Moreover, since X is integral, we can verify that the sequence

$$0 \to \bigoplus_{i=1}^{t} \mathscr{O}_X(-m_i) \xrightarrow{\varepsilon} \mathscr{M} \xrightarrow{\rho} \mathscr{I}_Y(c) \to 0$$

is exact with the use of the exactness of (2.4.1) and the torsion-freeness of $\mathscr{M}$. □

We need the following property that one can prove applying the local version of [12, Corollary 1.20] to the graded case.

Lemma 2.5. *Let $I \in R$ be a homogeneous ideal of height $p \geq 2$ and M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}_R^i(M,R)=0$ for $1 \leq i \leq p-1$. Assume that they fit into an exact sequence of the form*

$$0 \to S_{p-1} \to S_{p-2} \to \cdots \to S_1 \to S_0 \oplus M \to I(c) \to 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free modules over R. Then R/I is equidimensional if and only if M is reflexive.

Lemma 2.6. *Let p be an integers with $2 \leq p \leq r$ and M a finitely generated torsion-free graded module over R with no free direct summand satisfying $\mathrm{Ext}_R^i(M,R)= 0$ for $1 \leq i \leq p-1$. Then, M is reflexive if and only if $\mathrm{cpx}(M)_\bullet$ is exact at $\mathrm{cpx}(M)_{-1}$. Moreover, $H_{\mathfrak{m}}^1(M)=0$ in this case.*

Proof. Let $F_\bullet := \mathrm{cpx}(M)_\bullet$ and a_0 be a negative integer such that $F_i=0$ for all $i < a_0$. As mentioned in the proof of Theorem 2.2, $F_\bullet$ is exact at F_0, so that we obtain a complex

$$(2.6.1) \qquad 0 \to M = \mathrm{Coker}(\partial_1^F) \to F_{-1} \xrightarrow{\partial_{-1}^F} F_{-2} \xrightarrow{\partial_{-2}^F} \cdots \xrightarrow{\partial_{a_0-1}^F} F_{a_0} \to 0$$

which is exact at M. Suppose that $F_\bullet$ is exact at F_{-1}, too. Then M is reflexive as is known well. Conversely, suppose that M is reflexive. Since $M^{\vee\vee} \cong M$ by assumption, we obtain two exact sequences

$$\cdots \to F_1 \xrightarrow{\partial_1^F} F_0 \to M \to 0,$$
$$0 \to M \cong M^{\vee\vee} \to F_{-1} \xrightarrow{\partial_{-1}^F} \mathrm{Im}(\partial_{-1}^F) \to 0$$

by the construction of $F_\bullet$. Hence $H_{-1}(F_\bullet)=0$. Moreover, it follows from the second sequence that $H_{\mathfrak{m}}^1(M) \cong H_{\mathfrak{m}}^0(\mathrm{Im}(\partial_{-1}^F))=0$. □

Lemma 2.7. *Let E be a finitely generated graded module over A and let $g_1,\ldots,g_q$ be homogeneous elements of A which form an E-regular sequence. Denoting the complement of $\mathrm{Spec}(A/(g_i))$ in $\mathrm{Spec}(A)$ by W_i for each $1 \leq i \leq q$, let $\check{C}^\bullet(\mathfrak{W},\mathscr{E})$ be the Čech complex of the coherent sheaf $\mathscr{E}$ on $W := \bigcup_{i=1}^q W_i$ defined by E with respect to the covering $\mathfrak{W} := (W_i)_{1\leq i\leq q}$. Suppose $q \geq 2$. Then $H^i(\check{C}^\bullet(\mathfrak{W},\mathscr{E}))=0$ for $0 < i < q-1$ and $H^0(\check{C}^\bullet(\mathfrak{W},\mathscr{E}))=E$.*

Proof. Let $\check{C}^\bullet := \check{C}^\bullet(\mathfrak{W}, \mathscr{E})$. Since $g_1, \ldots, g_q$ are homogeneous and

$$\check{C}^l = \bigoplus_{1 \le i_1 < \cdots < i_{l+1} \le q} \mathscr{E}(W_{i_1} \cap \cdots \cap W_{i_{l+1}}) = E_{g_{i_1} \ldots g_{i_{l+1}}},$$

the module $\check{C}^{l\bullet}$ is naturally graded for all $0 \le l \le q-1$. For each $t \in \mathbf{N}$, let $K^\bullet_{(t)} := K^\bullet(g_1^t, \ldots, g_q^t; A)$ be the dual of the Koszul complex of $g_1^t, \ldots, g_q^t$ with respect to A and $K^\bullet_{(t)}(E) := K^\bullet_{(t)} \otimes_A E$. We have

$$K^l_{(t)}(E) = \bigoplus_{1 \le i_1 < \cdots < i_l \le q} (\chi_{(t)i_1})^* \wedge \cdots \wedge (\chi_{(t)i_l})^* \otimes E,$$

where $(\chi_{(t)i})^*$ denotes the free basis of $K^\bullet_{(t)}$ associated with g_i^t. Further, for each pair $s, t \in \mathbf{N}$ with $s \le t$, let $\xi^\bullet_{(s,t)} : K^\bullet_{(s)} \to K^\bullet_{(t)}$ be the chain map such that

$$\xi^l_{(s,t)}((\chi_{(s)i_1})^* \wedge \cdots \wedge (\chi_{(s)i_l})^*) = (g_{i_1} \ldots g_{i_l})^{t-s} (\chi_{(t)i_1})^* \wedge \cdots \wedge (\chi_{(t)i_l})^*.$$

Then the homomorphism $\xi^l_{(s,t)}$ is homogeneous of degree zero for all l, s, t. Moreover, $\{K^\bullet_{(t)}(E), \xi^\bullet_{(s,t)} \otimes E\}_{s,t \in \mathbf{N}}$ forms a direct system of complexes. Let

$$K^\bullet_\infty(E) := \varinjlim_t K_{(t)}{}^\bullet(E).$$

By the same argument as in the proof of [6, Proposition 3.5.5], we can construct homogeneous isomorphisms $\theta^i : K^{i+1}_\infty(E) \to \check{C}^i$ of degree zero for $0 \le i \le q-1$ compatible with the differentials of $K^\bullet_\infty(E)$ and $\check{C}^\bullet$. Since the elements $g_1, \ldots, g_q$ form an E-regular sequence by hypotheses, $H^i(K^\bullet_{(t)}) = 0$ for all $t \in \mathbf{N}$ and $i < q$. Hence, $H^i(\check{C}^\bullet) \cong H^{i+1}(K^\bullet_\infty) = 0$ for $0 < i < q-1$ and $H^0(\check{C}^\bullet) \cong \mathrm{Ker}(\partial^1_\infty) \cong K^0_\infty = E$, where $\partial^1_\infty : K^1_\infty \to K^2_\infty$ denotes the first differential of $K^\bullet_\infty$. □

Theorem 2.8. *Let p be an integer with $2 \le p \le r-2$ and M a finitely generated torsion-free graded reflexive module over R with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $1 \le i \le p-1$. Then, there is a homogeneous prime ideal I of height p which fits into an exact sequence of the form*

$$0 \to S_{p-1} \to S_{p-2} \to \cdots \to S_1 \to S_0 \oplus M \to I(c) \to 0, \tag{2.8.1}$$

where c is an integer and S_i $(0 \le i \le p-1)$ are finitely generated graded free modules over R.

Proof. Let $F_\bullet := \mathrm{cpx}(M)_\bullet$ and a_0 be a negative integer such that $F_i = 0$ for all $i < a_0$. Then $F_\bullet$ is exact at F_{-1} by Lemma 2.6 as well as at F_0, since M is reflexive. Hence there are exact sequences

$$\begin{cases} 0 \to M \to F_{-1} \to \mathrm{Im}(\partial^F_{-1}) \to 0, \\ 0 \to \mathrm{Im}(\partial^F_{-1}) \to F_{-2}. \end{cases} \tag{2.8.2}$$

Suppose $F_{-2} = \bigoplus_{l=1}^b R(-c_i)$ and let $c' := \min(c_1, \ldots, c_b)$. Then

$$[F_{-2}]_l = 0 \quad \text{for all } l < c'. \tag{2.8.3}$$

Since $H^i(F_\bullet^\vee)=0$ for $i\le 0$ by our construction and $H^i(F_\bullet^\vee)=\mathrm{Ext}_R^i(M,P)=0$ for $1\le i\le p-1$, we see $\dim(H_i(F_\bullet))=\dim(\mathrm{Ext}_R^{p-i}(\mathrm{Coker}(\partial_p^{F\vee}),R))\le r-p+i$ for all $i<p$. Moreover, $H_{-1}(F_\bullet)=0$ as mentioned above. Thus, $\dim(H_i(F_\bullet))\le r-p-2$ for all $a_0\le i\le -1$. Let $\mathfrak{a}$ be the product of $\mathrm{ann}(H_i(F_\bullet))$ $(a_0\le i\le -1)$. Then, for every homogeneous prime ideal $\mathfrak{p}\subset R$ not containing $\mathfrak{a}$, we have $H_i(F_\bullet)_\mathfrak{p}=0$ for all $a_0\le i\le -1$, so that $M_\mathfrak{p}$ is free. Besides, $\mathrm{ht}(\mathfrak{a})\ge p+2$. By Theorem 2.2, there are a homogeneous R-regular sequence $f_1,\dots,f_{p-2}$ with $f_i\in\mathfrak{a}$ for all $1\le i\le p-2$, a finitely generated graded module $\tilde{M}$ over $\bar{R}:=R/(f_1,\dots,f_{p-2})$, and a homomorphism $\varphi:M\to\tilde{M}$ over R satisfying the conditions (i), (ii), (iii), (iv) stated there with $u=p+2$. Let $\tilde{\mathscr{M}}$ be the torsion-free coherent sheaf on $X:=\mathrm{Proj}(\bar{R})$ defined by $\tilde{M}$. Then X is an integral normal scheme, and there is a subscheme Z of codimension 4 in X such that, in the outside of Z, X is smooth and $\tilde{\mathscr{M}}$ is locally free and orientable.

Suppose that $\mathrm{rank}_{\mathscr{O}_X}(\tilde{\mathscr{M}})\ge 2$. By Lemma 2.4, there are integers m_i $(1\le i\le t)$, n and a two-codimensional closed subscheme Y of X smooth in the outside of a subscheme, say Z', of X of codimension not less than $\min(4,6)=4$, whose ideal sheaf $\tilde{\mathscr{I}}_Y$ fits into an exact sequence of the form

$$0\to\bigoplus_{i=1}^{t}\mathscr{O}_X(-m_i)\to\tilde{\mathscr{M}}\to\tilde{\mathscr{I}}_Y(c)\to 0$$

with $c=n+\sum_{i=1}^{t}m_i$. Here we may assume with no loss of generality that $c>-c'$. Let $\tilde{I}$ be the saturated homogeneous ideal of Y in $\bar{R}$. Then, since $\tilde{M}$ is torsion-free and $H_\mathfrak{m}^1(\tilde{M})\cong H_\mathfrak{m}^1(M)=0$ by Lemma 2.6 and (i) of Theorem 2.2, we have a Bourbaki sequence

$$0\to\bigoplus_{i=1}^{t}\bar{R}(-m_i)\to\tilde{M}\to\tilde{I}(c)\to 0.$$

By Lemma 2.3, there is a homogeneous ideal I in R of height p containing $f_1,\dots,f_{p-2}$ satisfying $I/(f_1,\dots,f_{p-2})=\tilde{I}$ and a homomorphism $\psi:M\to I(c)$ such that $H_\mathfrak{m}^i(\psi):H_\mathfrak{m}^i(M)\to H_\mathfrak{m}^i(I(c))$ is an isomorphism for all $i\le r-p$. There are a finitely generated graded free module S_0 and a surjective homomorphism $\eta:S_0\oplus M\to I(c)$ such that $\eta|_M=\psi$. Let $K:=\mathrm{Ker}(\eta)$. Since $H_\mathfrak{m}^i(S_0)=0$ for all $i<r$, the homomorphisms $H_\mathfrak{m}^i(\eta):H_\mathfrak{m}^i(S_0\oplus M)\to H_\mathfrak{m}^i(I(c))$ $(0\le i\le r-p)$ are isomorphisms. Besides, $H_\mathfrak{m}^i(S_0\oplus M)=H_\mathfrak{m}^i(M)=0$ for $r-p+1\le i\le r-1$ by local duality, and $H_\mathfrak{m}^{r-p+1}(I)\ne 0$ since $\dim(R/I)=r-p$. It follows from the exact sequence

(2.8.4) $\quad 0\to K\to S_0\oplus M\to I(c)\to 0$

that $H_\mathfrak{m}^i(K)=0$ for all $0\le i\le r-p+1$ and that $H_\mathfrak{m}^{r-p+2}(K)\ne 0$. Hence, K has a free resolution

(2.8.5) $\quad 0\to S_{p-1}\to S_{p-2}\to\cdots\to S_1\to K\to 0$

over R of length $p-2$. Joining the above two exact sequences, we obtain an exact sequence of the form (2.8.1). Since $H_\mathfrak{m}^1(M)=0$, it follows from the long exact sequence of local cohomologies arising from (2.8.4) that $H_\mathfrak{m}^1(I)=0$, therefore I is saturated.

Moreover, since M is reflexive, the ring R/I is equidimensional by Lemma 2.5. Thus $Y = \mathrm{Proj}(R/I)$ is an equidimensional closed subscheme of $\mathbf{P} := \mathrm{Proj}(R)$ of codimension p which is smooth in the outside of Z'. In particular, $(R/I)_{\mathfrak{p}/I}$ is a field for every prime ideal $\mathfrak{p}$ in R of height p containing I. Let $\mathfrak{p}$ be a prime ideal in R of height larger than or equal to $p+1$ containing I. Then, computing local cohomologies after localization of the sequences (2.8.2), (2.8.4) and (2.8.5) by $\mathfrak{p}$, we find that $H^1_{\mathfrak{p}}(I_{\mathfrak{p}}) \cong H^1_{\mathfrak{p}}(M(-c)_{\mathfrak{p}}) = 0$. It follows, therefore, from the exact sequence $0 \to I_{\mathfrak{p}} \to R_{\mathfrak{p}} \to (R/I)_{\mathfrak{p}/I} \to 0$ that $H^0_{\mathfrak{p}/I}((R/I)_{\mathfrak{p}/I}) = 0$, namely, $\mathrm{depth}_{\mathfrak{p}/I}((R/I)_{\mathfrak{p}/I}) > 0$. This implies that R/I is reduced also (see e.g. [10, p. 125]).

Let $g_1, \ldots, g_{p+2}$ be homogeneous polynomials of R forming an R-regular sequence such that the complete intersection $Z'' := \mathrm{Proj}(R/(g_1, \ldots, g_{p+2}))$ contains Z'. Such $g_1, \ldots, g_{p+2}$ actually exist, since the codimension of Z' is not less than $p+2$ in $\mathbf{P}$. Let $\mathscr{I}$ be the ideal sheaf on $\mathbf{P}$ defined by I and $U'' := \mathbf{P} \setminus Z''$. We want to prove that $Y \cap U''$ is connected by showing $H^0(\mathcal{O}_{Y \cap U''}) \cong k$. Since, we have an exact sequence

$$0 \to H^0(\mathscr{I}|_{U''}) \to H^0(\mathcal{O}_{U''}) \to H^0(\mathcal{O}_{Y \cap U''}) \to H^1(\mathscr{I}|_{U''})$$

arising from the short exact sequence

$$0 \to \mathscr{I}|_{U''} \to \mathcal{O}_{U''} \to \mathcal{O}_{Y \cap U''} \to 0,$$

it is enough to verify for this purpose that

$$(2.8.6) \qquad H^0(\mathscr{I}|_{U''}) = 0,$$

$$(2.8.7) \qquad H^0(\mathcal{O}_{U''}) \cong k,$$

$$(2.8.8) \qquad H^1(\mathscr{I}|_{U''}) = 0.$$

Let $W_i := \mathrm{Spec}(R) \setminus \mathrm{Spec}(R/(g_i))$ $(1 \le i \le p+2)$ be the Zariski open subsets of $\mathrm{Spec}(R)$, $W := \bigcup_{i=1}^{p+2} W_i$, and $\mathfrak{W} := (W_i)_{1 \le i \le p+2}$ a covering of W. Using the Čech cohomology of $\mathcal{O}_W$ with respect to the covering $\mathfrak{W}$, we see $\oplus_{l \in \mathbf{Z}} H^i(\mathcal{O}_{U''}(l)) \cong H^i(\check{C}^\bullet(\mathfrak{W}, \mathcal{O}_W))$. Besides, $H^i(\check{C}^\bullet(\mathfrak{W}, \mathcal{O}_W)) = 0$ for $0 < i < p+1$ and $H^0(\check{C}^\bullet(\mathfrak{W}, \mathcal{O}_W)) = R$ by Lemma 2.7. Hence

$$(2.8.9) \qquad H^i(\mathcal{O}_{U''}(l)) = 0 \quad \text{for all } l \in \mathbf{Z},\ 0 < i < p+1,$$

$$(2.8.10) \qquad H^0(\mathcal{O}_{U''}(l)) \cong [R]_l \quad \text{for all } l \in \mathbf{Z}.$$

This proves (2.8.7). Let $\mathscr{M}$, $\mathscr{N}$, $\mathscr{F}_{-1}$, and $\mathscr{F}_{-2}$ be the coherent sheaves on $\mathbf{P}$ defined by M, $\mathrm{Im}(\partial^F_{-1})$, F_{-1}, and F_{-2}, respectively. Sheafifying the exact sequences (2.8.4) and (2.8.5) on U'', and then computing cohomologies, we find that $H^1(\mathscr{I}|_{U''}) \cong H^1(\mathscr{M}(-c)|_{U''})$. On the other hand, it follows from the exact sequences

$$H^0(\mathscr{N}(-c)|_{U''}) \to H^1(\mathscr{M}(-c)|_{U''}) \to H^1(\mathscr{F}_{-1}(-c)|_{U''}) \cong 0,$$

$$0 \to H^0(\mathscr{N}(-c)|_{U''}) \to H^0(\mathscr{F}_{-2}(-c)|_{U''})$$

arising from (2.8.2) that $H^1(\mathscr{M}(-c)|_{U''}) = 0$, since $H^0(\mathscr{F}_{-2}(-c)|_{U''}) = [F_{-2}]_{-c} = 0$ by (2.8.10) and (2.8.3). Hence (2.8.8) holds. Since $\dim(Y) > \dim(Z'')$, the set $Y \cap U''$ is not empty. The equality (2.8.6) follows from this and (2.8.7). Hence, $H^0(\mathcal{O}_{Y\cap U''})\cong k$. This means that $Y\cap U''$ is connected.

Now Y is an equidimensional reduced scheme such that $Y\cap U''$ is smooth and connected. Since each irreducible component of Y has a nonempty intersection with U'' by equidimensionality, Y must be integral. Hence, the saturated ideal I is prime.

In case $\mathrm{rank}_{\mathcal{O}_X}(\tilde{\mathcal{M}})<2$, replacing M, $\tilde{M}$, $\tilde{\mathcal{M}}$, and φ with $R^2\oplus M$, $\bar{R}^2\oplus\tilde{M}$, $\mathcal{O}_X^2\oplus\tilde{\mathcal{M}}$, and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \varphi \end{pmatrix},$$

respectively, we reach the same conclusion. □

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